(c) If the formula for erf(x) cannot be expressed in terms of more elementary functions, how is it possible for a subroutine to compute its values?

P1–2.–In 250 B.C.E., the Greek mathematician Archimedes estimated the number π as follows. He looked at a circle with diameter 1, and hence circumference π. Inside the circle he inscribed a square; see Figure P1.2. The perimeter of the square is smaller than the circumference of the circle, and so it is a lower bound for π. Archimedes then considered an inscribed octagon, 16-gon, etc., each time doubling the number of sides of the inscribed polygon, and producing ever better estimates for π. Using 96-sided inscribed and circumscribed polygons, he was able to show that $223/71 < \pi < 22/7$. There is a recursive formula for these estimates. Let p_n be the perimeter of the inscribed polygon with 2^n sides. Then p_2 is the perimeter of the inscribed square, $p_2 = 2\sqrt{2}$. In general

$$p_{n+1} = 2^n \sqrt{2(1 - \sqrt{1 - (p_n/2^n)^2})}$$

Compute p_n for $n = 3, 4, \ldots, 60$. Try to explain your results. (This problem was suggested by Alan Cline.)

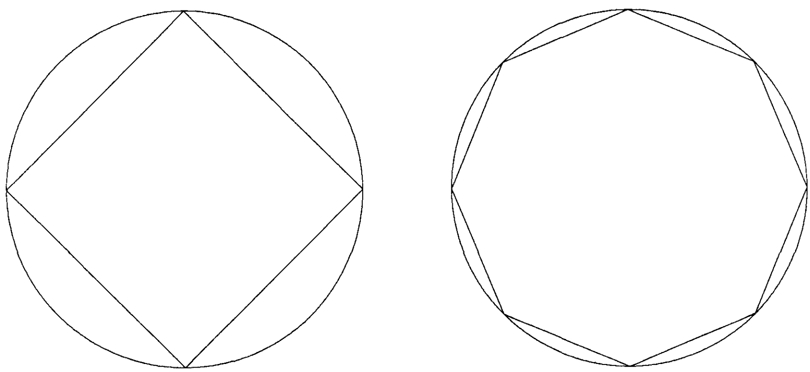

Figure P1.2

(a) Using this program try to compute $X(0.1, 200, 2000) \approx 0.03$. Comment on the difficulties you encounter.

(b) One approach is developed in the book by P. Sterbenz (1974). Let B be a large power of 2 chosen so that $2000B$ will not overflow and $B/2000$ will not underflow. For example $B = 2^{100}$. If X becomes larger than B we shall divide it by B. Then the true value we are computing is represented by $\mathrm{X}{\cdot}B$. If we use a counter I to count the number of times we divide by B the correct output is

$$X(p, k, N) = \mathrm{X} \cdot (B)^I, \qquad I \geq 0.$$

Similarly set $S = 1/B$. If X becomes smaller than S, multiply X by B and subtract one from I. At the conclusion of the computation if $I < 0$ we may be able to divide X by B I times without underflow and get a meaningful answer. Otherwise the best answer we can get is zero. Modify your program and try it on the problem in (a).

P2–7.–The formula derived in Problem P1–2 to estimate π,

$$p_{n+1} = 2^n \sqrt{2(1 - \sqrt{1 - (p_n/2^n)^2})},$$
$$p_2 = 2\sqrt{2},$$

fails due to a combination of underflow and catastrophic cancellation.

(a) Explain the failure in the formula.

(b) The formula can be improved so that the subtraction is eliminated. First write p_{n+1} as

$$p_{n+1} = 2^n \sqrt{r_{n+1}},$$

where

$$r_{n+1} = 2(1 - \sqrt{1 - (p_n/2^n)^2}), \qquad r_3 = 2/(2 + \sqrt{2}).$$

Show that

$$r_{n+1} = \frac{r_n}{2 + \sqrt{4 - r_n}}.$$

Use the last iteration to calculate r_n and p_n for $n = 3, 4, \ldots, 60$. (This revision was suggested by W. Kahan.)

(c) Eventually, $4 - r_n$ will round to 4, and so the formula derived in (b) is still affected by rounding errors for large values of n. Should this concern us?

P2–8.–Archimedes' method for estimating π can be derived in a different way. Let $P(h)$ denote the perimeter of the n-sided inscribed polygon, where $h = 1/n$.

(a) Using geometry, show that

$$P(h) = (1/h) \times \sin(\pi h).$$

Write a program to evaluate this function for $n = 3, 6, 12, 24, 48$ and 96. Print your estimate and the error for each value of n.